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Rules for calculation of approximate value of definite integral

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Abstract

We can calculate the definite integral exact value of f function with basic theorem of calculus which explains:

$$I = \int_a^b f(x)dx = F(a) - F(b)$$

If we can find an initial function for f. Despite all the conventional methods which exist for determining of initial function, we can find functions that are integrable in closed interval of $[a,b]$, but there is no specified rule for them to determine an initial function. For example we consider the definite integral of $\int_0^1 e^{-x^2}dx$, this integral exists but we can compute with classic rules. For approximate value determination these kinds of integrals there are some rules that we research three of them (trapezoid, Simpson, midpoint) in this article.

Keywords: Trapezoid rule, Simpson's rule, midpoint rule, error estimation

1. Introductions

If we cannot integrate definite integral with analytical rules, we can compute with numerical integration rules. We evaluate the I approximate value of definite integral with $f(x)$ values in $[a,b]$ number of finite point. The obtaining this kind of approximate is called numerical integration. We can use the up and down sums (Riemann's sum) for this purpose, but these sums usually require much more calculations than the rules mentioned here to obtain the desired accuracy. We need to calculate $f(x)$ in an equal space sum from $[a,b]$ points for calculation of definite integral in trapezoid, Simpson, midpoint rules. The calculation for determining an approximate value I integral are approximately proportional the number of function values required. In conclusion, to obtain a desired degree from integral, it is better method that it requires less computation function.

2. Trapezoid rule

The trapezoidal rule for calculation of definite integral is based on approximating between a curve and axis of x with the help of a trapezoid, it is fixed instead of rectangles. The length of the subintervals which obtains by $x_{n-1}, \ldots, x_2, x_1$, it is not necessary to be equal, but if it is equal the yield formula gets easier. Therefore, we suppose the length of each subinterval as follow:

$$h = \Delta x = \frac{b-a}{n}$$

2.2. Definition (Trapezoid rule)

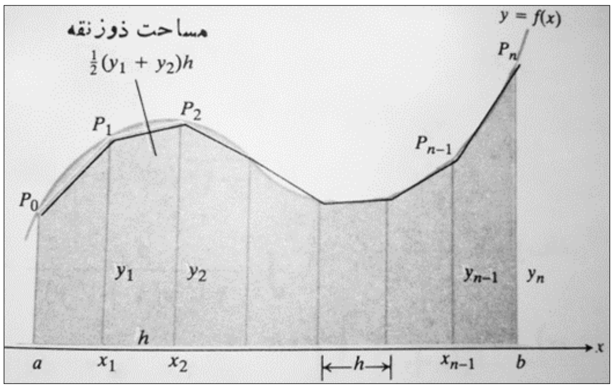
The $-n$ subinterval of trapezoid rule denoted with $\int_a^b f(x)dx$ and it is equal to

$$T_n = h(\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \cdots + y_{n-1} + \frac{1}{2}y_n)$$

(Nekokar & Darwishi, Numerical Calculation, 2013).
Now we explain the trapezoid rule by it the approximate of an integral that we know its value before

$$I = \int_1^2 \frac{1}{x}dx = \ln 2 = 0.69314718...$$

(This value and all mentioned approximates in this article were calculated with a scientific calculator.)



2.3. Figure: In trapezoidal rule we approximate the small curve lines to lines. For estimation of dark area, we sum the trapezoidal areas which are created by connecting two tips of these lines to x axis. Hence, we can approximate the main integral of $\int_a^b f(x)dx$ by sum of these trapezoidal areas (Thomas & Fini, 2013) [9].

2.4 Example: Compute the T_8, T_4 and T_{16} trapezoid rule approximates for

Solution: We have $h = \frac{(2-1)}{4} = \frac{1}{4}$ for $n = 4$; $h = \frac{1}{8}$ for $n = 8$; $h = \frac{1}{16}$ for $n = 16$

Therefore,

$$T_4 = \frac{1}{4}\left[\frac{1}{2}(1) + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\left(\frac{1}{2}\right)\right] = 0.69702381...$$

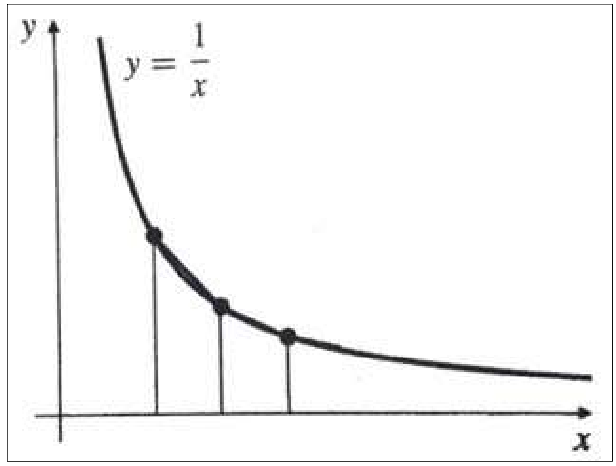
$$T_8 = \frac{1}{8}\left[\frac{1}{2}(1) + \frac{8}{9} + \frac{4}{5} + \frac{8}{11} + \frac{2}{3} + \frac{8}{13} + \frac{4}{7} + \frac{8}{15} + \frac{1}{2}\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{8}\left[4T_4 + \frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right] = 0.69412185...$$

$$T_{16} = \frac{1}{16}\left[8T_8 + \frac{16}{17} + \frac{16}{19} + \frac{16}{21} + \frac{16}{23} + \frac{16}{25} + \frac{16}{27} + \frac{16}{29} + \frac{16}{31}\right]$$

$$= 0.69339120...$$

Focus on how function values are applied in calculation of T_4, it was reapplied in T_8. Hence, how function values relate to T_8 into T_{16} were applied. When approximate requires, the doubling number of subintervals for every new calculation is very effective.



2.5. Figure
If the curve be on concave, the trapezoid areas are maximum than under curve area.
In conclusion the values of f which are already calculated, can be applied again.

All the trapezoid rule approximates to $I = \int_1^2 \frac{1}{x}dx$ are maximum from the exact value of I. That's why the graph of $y = \frac{1}{x}$ by $[1,2]$ is above the concave and therefore, the above section of approximate trapezoids are on curve.
The accurate errors can be calculated in three approximates, because we know that $\int_1^2 \frac{1}{x}dx = \ln 2 = 0.69314718...$
(Remember that the error in approximate is always equal to the exact negative value of the approximate value.)

$$I - T_4 = 0.69314718... - 0.6902381...$$

$$= 0.00387663...$$

$$I - T_8 = 0.69314718... - 0.69412185... = -0.00097467...$$

$$I - T_{16} = 0.69314718... - 0.69339120... = 0.00024402...$$

Every time that n doubles, the error reduces about one-fourth of its previous value. In the following we will show that it is possible for function like $\frac{1}{x}$.

2.6. Example: It is in an artificial way, it means that we know the exact value of integral, in this case we do not need to an approximate. In practical application of numerical integrating, we do not know the exact value we are tempted to compute some approximate for ascending values of n until the two most recent agree on a maximum acceptable value of prescribe error. We may claim that $In2 \approx 0.69...$

from the comparison of T_4 to T_8, and with more comparison we may suggest T_{16} and T_8 that the third decimal number is about 3: $I \approx 0.693...$. Although, this approach cannot be legitimized in general, it is often used in practice.

3. Error estimation in trapezoid rule

If f function in closed interval of $[a,b]$ includes second continuous derivation and M be a top bound for $|f''|$ values, in this case E_T error in f approximate integral from a to b satisfies for the following inequality (Silverman, 2009) [8].

$$|E_T| \le \frac{b-a}{12} h^2 M$$

3.1. Example: Set bound for the corresponding errors of T_8, T_4, T_{16} and for $I = \int_1^2 \frac{1}{x} dx$.

Solution: If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$. In $[1,2]$ interval we have $|f''(x)| \le 2$. In conclusion we can suppose in estimate $M = 2$, so

$$|E_T| \le \frac{2(2-1)}{12}\left(\frac{1}{4}\right)^2 = 0.0104....,$$

$$|E_T| \le \frac{2(2-1)}{12}\left(\frac{1}{8}\right)^2 = 0.026....,$$

$$|E_T| \le \frac{2(2-1)}{12}\left(\frac{1}{16}\right)^2 = 0.00065....,$$

The previous exact calculated errors are smaller than these bounds, since $|f''(x)|$ over most $[1,2]$ interval is relatively smaller than $M = 2$.

Comment: Error bounds often do not obtain easily like example 2. Specially, if the actual formula for $f(x)$ not be specified. Then we do not have any approach for calculation of $f''(x)$; in conclusion we cannot set M. The above inequality theoretical significance is more than its practical.

Simpson's rule

The other approach that we use for computation of definite integral approximate value is Simpson Rule (Parabola). In this approach better approximate obtains than trapezoid approximate. In this approach like trapezoid rule we obtain $p_0, p_1, ..., p_n$ points, but instead of connecting points with straight lines, we connect them with parabola line to each other. Before discussing more about its details, first we prove the following theorem.

3.2. Theorem

The bounded surface between $y = ax^2 + bx + c$ parabola, x axis, $x = x_0$ and $x = x_2$ lines and y axis. If $x_2 - x_0 = 2h$ then it is equal to

$$s = \frac{1}{3} h\left(y_0 + 4y_1 + y_2\right)$$

In which,

$x_1 = x_0 + h$ $y_2 = y(x_2)$, $y_1 = y(x_1)$, $y_0 = y(x_0)$

Proof:

$$y_0 = ax_0^2 + bx_0 + c$$

$$y_1 = a\left(x_0 + h\right)^2 + b\left(x_0 + h\right) + c$$

$$y_2 = a\left(x_0 + 2h\right)^2 + b\left(x_0 + 2h\right) + c$$

And in conclusion

$$y_0 + 4y_1 + y_2 = a\left(6x_0^2 + 12hx_0 + 8h\right) + b\left(6x_0 + 6h\right) + 6c$$

On the other hand,

$$s = \int_x^{x+2x} \left(ax^2 + bx + c\right) dx = \frac{a}{3} x^3 + \frac{b}{2} x^2 + cx \;\Big]_x^{x+2h}$$

$$= \frac{h}{3}\left(a\left(6x_0^2 + 12hx_0 + 8h\right) + b\left(6x_0 + 6h\right) + 6c\right)$$

By consideration of above,

$$s = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Suppose that f is connected in $[a,b]$ closed interval, we divide the $[a,b]$ interval into $2n$ equal parts to obtain $x_0, x_1, ..., x_{2n}$ points. From these points we draw the y s axis to intersect the function graph in $p_0, p_1, ..., p_{2n}$ points. We know that a parabola passes through three unreal points on an endurance. We pass $y = ax^2 + bx + c$ equation from p_2, p_1, p_0 points on a parabola graph. With consideration to previous theorem we can say the area between parabola and $x = x_0$ and $x = x_2$, the x es axis is equal to

$$s_1 = \frac{1}{3} h\left(y_0 + 4y_1 + y_2\right)$$

In which $h = \frac{b-a}{2n}$, on the other hand this area is equal to the region area between function graph, $x = x_0$, $x = x_2$ lines and x es axis, thus

$$\int_{x_0}^{x_2} f(x) dx \approx \frac{1}{3} h\left(y_0 + 4y_1 + y_2\right)$$

Likewise, if we consider the region between function graph of f, $x = x_4$, $x = x_2$ lines and x es axis, then we have

$$\int_{x_2}^{x_4} f(x)dx = \frac{1}{3}h(y_2 + 4y_3 + y_4)$$

$$\int_{x_{2n-2}}^{x_{2n}} f(x)dx \approx \frac{1}{3}h(y_{2n-2} + 4y_{2n-1} + y_{2n})$$

With sum of above relation, we have

$$\int_a^b f(x)dx \approx \frac{1}{3}h(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + \ldots + y_{2n-2} + 4y_{2n-1} + y_{2n})$$

Or

$$\int_a^b f(x)dx \approx \frac{1}{3}h\left(y_0 + y_{2n} + 4(y_1 + y_3 + \ldots + y_{2n-1}) + 2(y_2 + y_4 + \ldots + y_{2n-2})\right)$$

(Nekokar, Calculus, 2007)

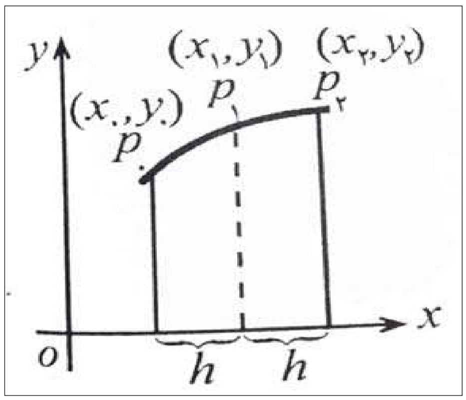


Fig 1.3

3.3. Definition (Simpson rule)

We divide $[a,b]$ interval into $2n$ equal -space in which $h = \frac{b-a}{2n}$, it was denoted with S_{2n} and it is equal to

$$\int_a^b f(x)dx \approx S_{2n} = \frac{b-a}{6n}\left(y_0 + y_{2n} + 4(y_1 + y_3 + \ldots + y_{2n-1}) + 2(y_2 + y_4 + \ldots + y_{2n-2})\right)$$

(Maroon, 2009)

3.4. Example: Compute S_8, S_4 and S_{16} approximates for $I = \int_1^2 \frac{1}{x}dx$, and compare them with $I = \ln 2 = 0.6914718\ldots$ actual value and also with T_8, T_4 and T_{16} yield in example 1.

Solution: We have

$$2n = 4$$

$$S_4 = \frac{1}{12}\left[1 + \frac{1}{2} + 4\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 4\left(\frac{4}{7}\right)\right]$$

$$= 0.69325397\ldots,$$

$$2n = 8$$

$$S_8 = \frac{1}{24}\left[1 + \frac{1}{2} + 4\left(\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right) + 2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right)\right]$$

$$= 0.69315453\ldots,$$

$$2n = 16$$

$$S_{16} = \frac{1}{48}\left[1+\frac{1}{2}+4\left(\frac{16}{17}+\frac{16}{19}+\frac{16}{21}+\frac{16}{23}+\frac{16}{25}+\frac{16}{27}+\frac{16}{29}+\frac{16}{31}\right)+2\left(\frac{8}{9}+\frac{4}{5}+\frac{8}{11}+\frac{2}{3}+\frac{8}{13}+\frac{4}{7}+\frac{8}{15}\right)\right]$$

$$0.69314765...$$

The three approximate errors are as follow:

$$I - S_4 = 0.69314718... - 0.69325397... = -0.00010679,$$

$$I - S_8 = 0.69314718... - 0.69315453... = -0.00000735,$$

$$I - S_{16} = 0.69314718... - 0.69314765... = -0.00000047,$$

These errors are obviously minor than trapezoid approximates (Babulian, Numerical analysis principle, 2013) [2, 3].

Error estimate in Simpson rule

If f function includes fourth continuous derivation $[a,b]$ bounded interval and N be a top bound for $|f''|$ values in this case the E_S error in f approximate integral from a to b satisfy the following inequality (Babulian, Numerical analysis, 2010) [2, 3].

$$|E_S| \le \frac{1}{180} h^2 (b-a) N$$

2.3. example into 1.3. approximates example, find bound for absolute errors.

Solution: If $f(x) = \frac{1}{x}$, thus

$$f'(x) = -\frac{1}{x^2},\ f''(x) = \frac{2}{x^3},$$

$$f^{(3)}(x) = -\frac{6}{x^4},\ f^{(4)}(x) = \frac{24}{x^5},$$

It is obvious that in $[1,2]$ we have $\left|f^{(4)}(x)\right| \le 24$; in conclusion for estimate $N = 24$ can be taken. We have

$$|E_S| \le \frac{24(2-1)}{180}\left(\frac{1}{4}\right)^4 \approx 0.00052083,$$

$$|E_S| \le \frac{24(2-1)}{180}\left(\frac{1}{8}\right)^4 \approx 0.00003255,$$

$$|E_S| \le \frac{24(2-1)}{180}\left(\frac{1}{16}\right)^4 \approx 0.00000203,$$

We remark again that actual errors are well in the dominance of these errors.

Low-order polynomial

If $f(x)$ be a polynomial which its order is less than four, in this case its fourth derivation is zero and we have

$$E_S = -\frac{b-a}{180} h^4 f^{(4)}(c) = -\frac{b-a}{180} h^4 (0) = 0.$$

therefore, in Simpson approximate every integral of f does not have any error. In other word, if the proved f be a first-order function (linear) or a second or third-order polynomial apart from number of subintervals in interval division, Simpson rule obtains the exact value of every f integral.

In addition, if there is proved f or first-order function (linear), thus its second derivation is zero and we have

$$E_T = -\frac{b-a}{12} h^2 f''(c) = -\frac{b-a}{12} h^2 (0) = 0.$$

Therefore, trapezoid rule obtains the exact value of every f integral.

3.5. Example: Compute the actual approximate $I = \int_0^2 x^2 dx$ integral by Simpson rule and compare it with exact value of integral.
Solution: We have

$$2n = 4$$

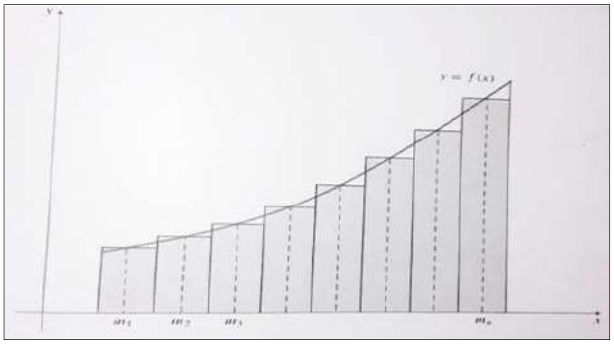
$$S_4 = \frac{1}{6}\left[0 + (2)^2 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2\right] = \frac{8}{3}$$

And the exact value of integral is

$$I = \int_0^2 x^2 dx = \frac{1}{3}x^3\Big]_0^2 = \frac{8}{3}$$

4. Midpoint rule

An approximate in simpler way to $\int_a^b f(x)dx$, based on $[a,b]$ partition to n equal-subinterval, requires to form a Riemann sum of rectangles areas which their heights in midpoints of n subinterval were taken.
The trapezoid and Simpson rules can compute integrals such as $\int_0^1 \frac{dx}{\sqrt{x}}$. These rules are required that $f(x)$ function be continuous at start and end points of the integral. The midpoint rule is not efficient for the computation of integrals which are not continuous at start and end point (Yousefi, 2010) [10].



4.1. Figure: The midpoint rule approximate of M_n to $\int_a^b f(x)dx$, Riemann sum based on heights to f graph is partition in subinterval points.

4.2 Definition: (Midpoint rule)

If $h = \frac{(b-a)}{n}$ as we let $1 \le j \le n$, so $m_j = a + \left(j - \left(\frac{1}{2}\right)\right)h$. The midpoint rule approximate to $\int_a^b f(x)dx$ denoted by M_n and it is equal to

$$M_n = h\left(f(m_1) + f(m_2) + ... + f(m_n)\right) = h\sum_{j=1}^{n} f(m_j).$$

4.3 Example: Find the midpoint rule approximates of M_4 and M_8 for $I = \int_1^2 \frac{1}{x}dx$ integral and compare their errors with obtained errors for trapezoid rule approximates.

Solution: To find M_4, we divide $[1,2]$ interval into four equal-subintervals $\left[\frac{3}{2},\frac{7}{4}\right],\left[\frac{5}{4},\frac{3}{2}\right],\left[1,\frac{5}{4}\right]$ and $\left[\frac{7}{4},2\right]$. The midpoints of these intervals are $,\frac{13}{8},\frac{11}{8},\frac{9}{8}$ and $\frac{15}{8}$. The midpoints of subintervals for M_8 obtain likewise. The required point rule approximates are

$$M_4 = \frac{1}{4}\left[\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right] = 0.69121989...$$

$$M_8 = \frac{1}{4}\left[\frac{16}{17} + \frac{16}{19} + \frac{16}{21} + \frac{16}{23} + \frac{16}{25} + \frac{16}{27} + \frac{16}{29} + \frac{16}{31}\right]$$

$$= 0.69266055...$$

The errors of these approximates are

$$I - M_4 = 0.69314718... - 0.69121989... = 0.00192729...$$
$$I - M_8 = 0.69314718... - 0.6926655... = 0.00048663...$$

These errors have different notations and they are about half of trapezoid rule.

4.4. Example: Obtain an approximate of $\int_0^{\frac{9}{100}} \frac{dx}{\sqrt{x}}$ by $h = \frac{3}{100}$ with midpoint rule.

Solution: We divide the $\left[0,\frac{9}{100}\right]$ interval into three equal-subintervals of $,\left[\frac{3}{100},\frac{3}{50}\right],\left[0,\frac{3}{100}\right]$ and $\left[\frac{3}{50},\frac{9}{100}\right]$. The midpoints of these intervals are $,\left[\frac{9}{200}\right],\left[\frac{3}{200}\right]$ and $\left[\frac{15}{200}\right]$.

$$M_3 = \frac{3}{100}\left[\sqrt{\frac{200}{3}} + \sqrt{\frac{200}{9}} + \sqrt{\frac{200}{15}}\right] = 4959$$

Error estimate in midpoint rule

If f function in closed interval of $[a,b]$ includes second continuous derivation and K be a top bound for $|f''|$ values. In this case, the E_M error in f approximate integral from a to b satisfies into the following inequality.

$$|E_M| \le \frac{b-a}{24}h^2K$$

5. Conclusion

Midpoint rule apparently is better than trapezoid because its error is about half than trapezoid rule error and the value of a function in point is calculated less. In addition, it is applicable for calculation of integrals in which the function at start and end of interval includes infinite integrating. But, trapezoid rule has marvelous specialty which midpoint and Simpson doesn't. In example 1.2 it was seen how applied function values in T_4 calculation reused in T_8 calculation, likewise how function corresponding values to T_8 were applied on T_{16}. But in Simpson rule polynomial functions which their order is less than four it doesn't have any error.

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